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# Exceptional Family of Elements and Feasibility for Nonlinear Complementarity Problems

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**Abstract.** In this paper, we introduce the new concept of $\alpha$-exceptional families of elements and $(\alpha, \beta)$-exceptional families of elements for continuous functions, and utilize these notions for the study of the feasibility of nonlinear complementarity problems in $R^n$ and an infinite-dimensional Hilbert space $H$ without the assumption $K^* \subseteq K$.

**Key words:** Exceptional family of elements, Nonlinear complementarity problem, Feasibility, Homotopy

## 1. Introduction

Complementarity problems have been applied in many areas such as traffic equilibrium, solution of the Nash equilibrium, Walrasian equilibrium model and prediction of inter-regional commodity flows, optimization, engineering, etc., which has advanced the study of complementarity problems, in finite-dimensional space studies extensively associated with the idea of equilibrium, in the past thirty years. Given a particular complementarity problem, its solvability is not evident, and so many existence theorems have been proved (see [1, 2, 4, 9]).

Due to the many applications to study the existence of solutions for nonlinear complementarity problems, it is very important to study the feasibility of nonlinear complementarity problems. A feasible but unsolvable nonlinear complementarity problem can be referred to in the literature [2, 6, 8]. Recently, a variety of concepts of exceptional families of elements (in short, EFE) for continuous functions were introduced, and some feasibility and existence theorems for nonlinear complementarity problems and variational inequalities were proved by many authors (see, for example, [3, 10–15] and the references therein).

Very recently, Isac [9] introduced the notion of an $(\alpha, \beta)$-exceptional family of elements for a continuous function, and applied this notion to the study of the feasibility of nonlinear complementarity problems, which can be considered as a new kind of EFE concept to the complementarity theory. More precisely, Isac proved the following results:

THEOREM A [9, Theorem 5.2]. *Let $(\alpha, \beta)$ be a pair of real numbers such that $0 \leqslant \alpha < \beta$, and let $K \subset R^n$ be a closed pointed convex cone such that $K^* \subset K$ or $K^* = K$. Then, for any continuous function $f : R^n \to R^n$, either problem*

*$NCP(f, K, R^n)$ is feasible or there exists an $(\alpha, \beta)$-exceptional family of elements for $f$ with respect to $K$.*

THEOREM B [9, Theorem 6.1]. *Let $(H, \langle, \rangle)$ be an infinite-demensional Hilbert space, let $K \subset H$ be a closed convex cone such that $K^* \subseteq K$, and let $f : H \to H$ be a completely continuous field of the form $f(x) = (1/\beta)x - T(x)$, where $\beta > 0$. Then, for any real $\alpha$ such that $0 \leqslant \alpha < \beta$, either problem $NCP(f, K, H)$ is feasible or there exists an $(\alpha, \beta)$-exceptional family of elements in sense of Definition 6.1 for $f$ with respect to $K$.*

At the end of the paper [9], Isac proposed three open problems, and one of them is whether Theorem A and Theorem B are true without the assumption $K^* \subseteq K$.

In this paper, we introduce the new concept of $\alpha$-EFE and $(\alpha, \beta)$-EFE for continuous functions, and utilize these notions to the study of the feasibility of nonlinear complementarity problems in $R^n$ and an infinite-dimensional Hilbert space $H$ without the assumption $K^* \subseteq K$.

## 2. Preliminaries

In this paper, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and inner product of the Hilbert space $H$, respectively. Let $K$ be a nonempty closed convex subset of $H$ and $f : H \to H$ be a continuous mapping. The variational inequality problem (in short, $VI(f, K, H)$) is to find $x^* \in K$ such that

$$\langle f(x^*), x - x^*\rangle \geqslant 0, \quad \forall x \in K. \tag{2.1}$$

If $K$ is a closed pointed convex cone in $H$, i.e, $K$ satisfies the following conditions:

$$(i)\ K + K \subset K; \quad (ii)\ \lambda K \subseteq K, \quad \forall\lambda \in [0, +\infty); \quad (iii)\ K \cap (-K) = \{0\},$$

then the problem (2.1) reduces to the following nonlinear complementarity problem (in short, $NCP(f, K, H)$): finding $x^* \in K$ such that

$$x^* \in K,\ f(x^*) \in K^*,\ \langle f(x^*), x^*\rangle = 0,$$

where $K^*$ is the dual cone of $K$, i.e.,

$$K^* = \{f \in R^n : \langle f, x\rangle \geqslant 0, \quad \forall x \in K\}.$$

REMARK 2.1. In optimal theory and practical problems, $K$ can be commonly formulated as follows

$$K = \{x \in R^n : g_i(x) \leqslant 0,\ i = 1, \cdots, m;\ h_j(x) = 0,\ j = 1, \cdots, l\},$$

where $g_i : R^n \to R^n$ is a continuously differentiable convex function and $h_j : R^n \to R^n$ is a affine function, $i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, l$.

DEFINITION 2.1. $NCP(f, K, H)$ is said to be feasible if $\{x \in K : f(x) \in K^*\} \neq \emptyset$ and $NCP(f, K, H)$ is called to be strictly feasible if $\{x \in K : f(x) \in intK^*\} \neq \emptyset$, where $intA$ denotes the set of all the interior of $A$.

Let $H$ be a Hilbert space and $K \subset H$ a closed pointed convex cone. Then for each $x \in H$, the projection $P_K(x)$ of $x$ on $K$ is characterized by the properties below:

$(P_1)$ $\langle P_K(x) - x, y\rangle \geqslant 0, \quad \forall y \in K,$
$(P_2)$ $\langle P_K(x) - x, P_K(x)\rangle = 0.$

LEMMA 2.1. *[5]. Let $\Omega \subset R^n$ be a bounded open set and $f : \overline{\Omega} \to R^n$ be a continuous mapping. Suppose $0 \notin f(\partial\Omega)$, where $\partial\Omega$ denotes the boundary of $\Omega$. There then exists an integer* $\deg(f, \Omega, 0)$, *satisfying the following properties:*

(i) $\deg(I, \Omega, 0) = 1 \Leftrightarrow 0 \in \Omega$, *where $I$ denotes the identity mapping;*
(ii) *If* $\deg(f, \Omega, 0) \neq 0$, *then $f(x) = 0$ has a solution in $\Omega$;*
(iii) *If $h : \overline{\Omega} \times [0, 1] \to R^n$ is continuous, and for each $(x, t) \in \partial\Omega \times [0, 1]$, $h(x, t) \neq 0$, then* $\deg(h(\cdot, t), \Omega, 0)$ *does not depend on $t$.*

DEFINITION 2.2 ([9, 12]) Let $K$ be a closed pointed convex cone in $R^n$. The sequence $\{x^r\}_{r>0} \subset K$ is called an exceptional family of elements for $f$ with respect to $K$, if $\|x^r\| \to +\infty$ as $r \to \infty$ and for each $r > 0$, there exists $t_r > 0$ such that

$$f(x^r) + t_r x^r \in K^* \quad \text{and} \quad \langle f(x^r) + t_r x^r, x^r\rangle = 0.$$

REMARK 2.2 *Definition 2.2 is a relatively generalized concept of an exceptional family of elements and unifies the corresponding definition in [11]. When $K = R^n_+$, it reduces to Definition 2.1 of Isac and Obuchowska [11], i.e., if $\|x^r\| \to +\infty$ as $r \to \infty$, and for each $r > 0$ there exists $t_r > 0$ such that for each $i = 1, 2, \cdots, n$,*

(i) $f_i(x^r) = -t_r x_i^r$, *if* $x_i^r > 0$
(ii) $f_i(x^r) \geqslant 0$, *if* $x_i^r = 0$.

## 3. Main Results

We first introduce a new concept of EFE.

DEFINITION 3.1. Let $K$ be a closed pointed convex cone in $R^n$ and $\alpha \geqslant 0$. The sequence $\{x^r\}_{r>0} \subset K$ is said to be an $\alpha$-exceptional family of elements for $f$ with respect to $K$, if $\|x^r\| \to +\infty (r \to \infty)$ and for each $r > 0$, there exists $\eta_r > 0$ such that

$$(i)\ \eta_r x^r + f(x^r) \in K^*, \quad (ii)\ \langle \eta_r x^r + f(x^r), (1 + \eta_r)x^r - \alpha P_K(f(x^r))\rangle = 0.$$

REMARK 3.1. If $\alpha = 0$, then Definition 3.1 reduces to Definition 2.2.

LEMMA 3.1 *([9]) Let $K \subset R^n$ be a closed pointed convex cone in $R^n$ and $f : R^n \to R^n$ be a continuous function. Then there exists either a solution for problem $NCP(f, K, R^n)$ or an exceptional family of elements for $f$ with respect to $K$.*

THEOREM 3.1 *Let $K \subset R^n$ be a closed pointed convex cone in $R^n$ and $f : R^n \to R^n$ be a continuous function. Then either $NCP(f, K, R^n)$ is feasible, or for each $\alpha \geqslant 0$, there exists an $\alpha$-exceptional family of elements with respect to $f$.*

*Proof.* For any $\alpha \geqslant 0$ and $r > 0$, let

$$B_r = \{x \in R^n : \|x\| < r\}, \quad \partial B_r = \{x \in R^n : \|x^r\| = r\}.$$
$$g(x) = \alpha P_K(f(x)) + P_K[x - f(x) - \alpha P_K(f(x))].$$

Since $f$ and the projection $P_K$ are continuous, we know that $g : R^n \to R^n$ is continuous. Define $h : R^n \times [0, 1] \to R^n$ as follows:

$$h(x, t) = x - tg(x).$$

It is easy to see that $h : R^n \times [0, 1] \to R^n$ is continuous and $h(x, 0) = x$ for each $x \in R^n$. For each $r > 0$, $0 \in B_r$ and $0 \notin \partial B_r$. It follows from (1) of Lemma 2.1 that

$$\deg(h(\cdot, 0), B_r, 0) = 1. \tag{3.1}$$

Suppose that $NCP(f, K)$ is infeasible. We now prove that for each $r > 0$, there exist $x^r \in \partial B_r$ and $t_r \in [0, 1]$ such that $0 = h(x^r, t_r)$. In fact, if there exists some $r > 0$ such that

$$h(x^r, t_r) \neq 0, \quad \forall x^r \in \partial B_r, t_r \in [0, 1],$$

it then follows from (3) of Lemma 2.1 that

$$\deg(h(\cdot, 0), B_r, 0) = \deg(h(\cdot, 1), B_r, 0). \tag{3.2}$$

From (3.1), (3.2) and (2) of Lemma 2.1, we know that

$$h(x, 1) = x - g(x) = 0$$

has a solution in $B_r$. Therefore, there exists $x^* \in B_r$ such that

$$x^* - \alpha P_K(f(x^*)) - P_K[x^* - f(x^*) - \alpha P_K(f(x^*))] = 0. \tag{3.3}$$

By the property $(P_1)$ of the projection $P_K$, we have

$$\langle x^* - \alpha P_K(f(x^*)) - [x^* - f(x^*) - \alpha P_K(f(x^*))], y\rangle \geqslant 0, \quad \forall y \in K,$$

and so

$$\langle f(x^*), y\rangle \geqslant 0, \quad \forall y \in K. \tag{3.4}$$

It follows from (3.3) and (3.4) that $x^* \in K$ and $f(x^*) \in K^*$. Thus, $NCP(f, K)$ is feasible, which is a contradiction. Therefore, for each $r > 0$, there exists $x^r \in \partial B_r$ and $t_r \in [0, 1]$ such that

$$x^r - t_r\{\alpha P_K(f(x^r)) + P_K[x^r - f(x^r) - \alpha P_K(f(x^r))]\} = 0. \tag{3.5}$$

Since $x^r \neq 0$, it follows that $t_r \neq 0$. We now prove $t_r \neq 1$. If $t_r = 1$, then

$$x^r - \alpha P_K(f(x^r)) - P_K[x^r - f(x^r) - \alpha P_K(f(x^r))] = 0. \tag{3.6}$$

The aforementioned proof shows that (3.6) implies $NCP(f, K, R^n)$ is feasible, which contradicts our assumption. Thus, $t_r \in (0, 1)$. In addition, it follows from (3.5) that

$$P_K[x^r - f(x^r) - \alpha P_K(f(x^r))] = \frac{1}{t_r}x^r - \alpha P_K(f(x^r)). \tag{3.7}$$

From (3.7), the properties $(P_1)$ and $(P_2)$ of the projection $P_K$, we have

$$\left\langle \frac{1}{t_r}x^r - \alpha P_K(f(x^r)) - [x^r - f(x^r) - \alpha P_K(f(x^r))], y \right\rangle \geqslant 0, \quad \forall y \in K,$$
$$\left\langle \frac{1}{t_r}x^r - \alpha P_K(f(x^r)) - [x^r - f(x^r) - \alpha P_K(f(x^r))], \frac{1}{t_r}x^r - \alpha P_K(f(x^r)) \right\rangle = 0,$$

that is,

$$\left\langle \left(\frac{1}{t_r} - 1\right) x^r + f(x^r), y \right\rangle \geqslant 0, \quad \forall y \in K, \tag{3.8}$$
$$\left\langle \left(\frac{1}{t_r} - 1\right) x^r + f(x^r), \frac{1}{t_r}x^r - \alpha P_K(f(x^r)) \right\rangle = 0. \tag{3.9}$$

Setting $\eta_r = \frac{1}{t_r} - 1$, then $\eta_r > 0$, and it follows from (3.8) and (3.9) that

$$\eta_r x^r + f(x^r) \in K^*, \tag{3.10}$$
$$\langle \eta_r x^r + f(x^r), (1 + \eta_r)x^r - \alpha P_K(f(x^r)) \rangle = 0. \tag{3.11}$$

Since $x^r \in \partial B_r$, we have $\|x^r\| = r$ and $\|x^r\| \to +\infty$ as $r \to +\infty$. It is easy to see that (3.7) implies $x^r \in K$. By (3.10), (3.11) and Definition 3.1, we know that $\{x^r\} \subset K$ is an $\alpha$-exceptional family of elements for $f$ with respect to $K$. This completes the proof. □

Combining Lemma 3.1 and Theorem 3.1, we have the following result:

THEOREM 3.2 *Let $K \subset R^n$ be a closed pointed convex cone in $R^n$ and $f : R^n \to R^n$ be a continuous function. If $NCP(f, K, R^n)$ is feasible but unsolvable, then there exists an exceptional family of elements for $f$ with respect to $K$, and there exists an $\alpha > 0$ such that there is no $\alpha$-exceptional family of elements for $f$ with respect to $K$.*

EXAMPLE 3.1 [9]. *Let $K$ be the nonnegative orthant of Euclidean space $(R^2, \langle\cdot,\cdot\rangle)$, i.e., $K = R^2_+$,*

$$f(x) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

*where $x = (x_1, x_2)^T \in R^2$, $T$ denotes the transposition. Obviously, $NCP(f, K, R^n)$ is feasible but unsolvable. For each $r > 0$, Let $x^r = (r, 0)^T \in R^2_+$. We know $\{x^r\}_{r>0}$ is an exceptional family of elements for $f$ with respect to $K$. In fact, letting $t_r = r + 2/r$ for $r > 0$, then $\|x^r\| \to +\infty$ and*

$$f(x^r) + t_r x^r \in K^*, \quad \langle f(x^r) + t_r x^r, x^r\rangle = 0.$$

*In the following, we study the case of infinite-dimensional Hilbert space $H$.*

DEFINITION 3.2. Let $K$ be a closed pointed convex cone in $H$ and $f : H \to H$ be continuous with the form of $f(x) = \frac{1}{\beta}x - T(x)$, where $\beta > 0$ and $T(x)$ is a completely continuous function. For each $0 \leqslant \alpha < \beta$, the sequence $\{x^r\}_{r>0} \subset K$ is said to be an $(\alpha, \beta)$-exceptional family of elements associated with $f$, if $\|x^r\| \to +\infty$ as $r \to \infty$, and for each $r > 0$, there exists $t_r > 0$ such that

$$(i)\ t_r x^r + \beta f(x^r) \in K^*, \quad (ii)\ \langle t_r x^r + \beta f(x^r), (1 + t_r)x^r - \alpha P_K[T(x^r)]\rangle = 0.$$

LEMMA 3.2. *Let $H$ be an arbitrary infinite-dimensional Hilbert space, $\Omega$ be a bounded open subset of $H$, $T : \overline{\Omega} \to H$ be completely continuous, and $0 \notin (I - T)(\partial\Omega)$. Then the Leray-Schauder degrees have the following properties:*

(i) $\deg(I, \Omega, 0) = 1 \Leftrightarrow 0 \in \Omega$;

(ii) *If* $\deg(I - T, \Omega, 0) \neq 0$, *then $Tx = x$ has a solution in $\Omega$;*

(iii) *If $h : [0, 1] \times \overline{\Omega} \to E$ is completely continuous, and for each $(t, x) \in [0, 1] \times \partial\Omega$, $T(t, x) \neq x$, then* $\deg(I - T(t, \cdot), \Omega, 0)$ *does not depend on $t$.*

THEOREM 3.3. *Let $K$ be a closed pointed convex cone in $H$. If $f : H \to H$ is continuous with the form of $f(x) = 1/\beta x - T(x)$, where $\beta > 0$, $T : H \to H$ is completely continuous, then either $NCP(f, K, H)$ is feasible, or for each $\alpha$ such that $0 \leqslant \alpha < \beta$, there exists an $(\alpha, \beta)$-exceptional family of elements for $f$ with respect to $K$.*

*Proof.* For any $r > 0$, let

$$\begin{aligned} B_r &= \{x \in H : \|x\| < r\}, \quad \partial B_r = \{x \in H : \|x\| = r\}, \\ g(x) &= \alpha P_K(T(x)) + P_K[x - \beta f(x) - \alpha P_K(T(x))]. \end{aligned}$$

The complete continuity of $T$ and the continuity of the projection of $P_K$ show that $g : H \to H$ is completely continuous. Let $h : [0, 1] \times H \to H$ be defined as follows

$$h(t, x) = tg(x).$$

Then $h$ is completely continuous on $[0, 1] \times H$. It is clear that, for each $r > 0$, $x - h(0, x) = 0$ has a solution in $B_r$. From (i) and (ii) of Lemma 3.2 that

$$\deg(I - h(0, \cdot), B_r, 0) \neq 0. \tag{3.12}$$

Suppose that $NCP(f, K, H)$ is infeasible. We can prove that for each $r > 0$, there exist $x^r \in \partial B_r$ and $t_r \in [0, 1]$ such that

$$x^r - h(t_r, x^r) = 0.$$

Indeed, if there exists $r > 0$ such that

$$x^r - h(t_r, x^r) \neq 0, \quad \forall x^r \in \partial B_r, t_r \in [0, 1],$$

then from Lemma 3.2, we have

$$\deg(I - h(1, \cdot), B_r, 0) = \deg(I - h(0, \cdot), B_r, 0) \neq 0.$$

Hence, $x - h(1, x) = 0$ is solvable in $B_r$, which implies that there exists $x^* \in B_r$ such that

$$x^* - \alpha P_K(T(x^*)) - P_K[x^* - \beta f(x^*) - \alpha P_K(T(x^*))] = 0. \tag{3.13}$$

Since $\beta > 0$, it follows from the property $(P_1)$ of the projection $P_K$ that

$$\langle f(x^*), y\rangle \geqslant 0, \quad \forall y \in K. \tag{3.14}$$

By (3.13), we have

$$x^* = \alpha P_K(T(x^*)) + P_K[x^* - \beta f(x^*) - \alpha P_K(T(x^*))] \in K. \tag{3.15}$$

It follows from (3.14) and (3.15) that $x^*$ is a feasible point of $NCP(f, K, H)$. It contradicts our assumption.

So for each $r > 0$, there exist $x^r \in \partial B_r$ and $t_r \in [0, 1]$ such that

$$x^r - t_r\{\alpha P_K(T(x^r)) + P_K[x^r - \beta f(x^r) - \alpha P_K(T(x^r))]\} = 0.$$

If $t_r = 0$, then $x^r = 0$. Since $x^r \in \partial B_r$, we know that $\|x^r\| = r$ and $t_r \neq 0$. If $t_r = 1$, then

$$x^r - \alpha P_K(T(x^r)) - P_K[x^r - \beta f(x^r) - \alpha P_K(T(x^r))] = 0. \tag{3.16}$$

Repeating the previous proof, we can prove $x^r$ is a feasible point of $NCP(f, K, H)$, which is a contradiction. Therefore, $t_r \in (0, 1)$ and it follows from (3.19) that

$$P_K[x^r - \beta f(x^r) - \alpha P_K(T(x^r))] = \frac{1}{t_r}x^r - \alpha P_K(T(x^r)).$$

By using the characterized properties $(P_1)$ and $(P_2)$ of the projection $P_K$ and similar proof of Theorem 3.1, we deduce immediately that $\{x^r\}_{r>0}$ is an $(\alpha, \beta)$-exceptional family of elements for $f$ with respect to $K$. This completes the proof. □

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